Quasi static finite element

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In solving the quasi-static linear elasticity problem, one relates strain to stress. How much should a collection of elements be strained to account for the stress related to the external forces?

When dealing with a planar simulation, there is no normal stress along the depth dimension, here z, and there is no shear stress in neither the xz nor the yz planes. The remaining stresses are then normal stress along x and y and shear stress in the xy plane.

$$\sigma_x = \frac{E}{1-\upsilon^2}(\varepsilon_x + \upsilon\varepsilon_y) \tag{1}$$

$$\sigma_y = \frac{E}{1-\upsilon^2}(\varepsilon_y + \upsilon\varepsilon_x) \tag{2}$$

$$\tau_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy} \tag{3}$$

E is modulus of elasticity, υ is Poisson ratio. Both are constants which can be looked up for a specific material. E relates strain to stress while υ relates strain along one axis to strain along an orthogonal axis, or how much the material thins (or in some cases grows) when being stretched.

Those three equations can be rewritten in matrix form as

$$\sigma = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\varepsilon \tag{4}$$

Where σ is the stress vector and ε is the strain vector.

In the above, only σ and ε are variable while the rest is constants for the material being used (assuming a single material). This means that it can be rewritten further by introducing D.

$$\sigma = D\varepsilon \tag{5}$$

Where

$$D = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \quad (6)$$

Knowing how stress relates to strain for an element, and thereby for a collection of interconnected elements, the next step will be to find the minimal potential energy for the system since the state with minimal potential energy is also the state of equilibrium where all forces balance eachother.

The strain energy u for a unit volume is

$$u = \frac{1}{2}\varepsilon^T \sigma \quad (7)$$

The reason it is just that is that the stress grows linearly with strain. Imagine a spring which is at rest length. It is then compressed some distance ε and now the resistance force, stress, is σ. The stress grew from 0 to σ over a distance of ε. This function was linear so it went linearly from 0 to σ. Integrating this linear development over the entire distance is $1\sigma\varepsilon$. In other words, the strain energy is the work done to strain this object considering the increasing stress.

To make this even more crystal clear. A linear spring has a stress force $F = -kx$ where x is the displacement from rest and k is the spring constant. Now integrate F over x, force times distance, to get the work done when compressing the spring. $\int -kxdx = -k\int xdx = -k\frac{1}{2}x^2$. This does not fit the equation above entirely since it just looks at the stress after displacement and then average this value and the initial zero stress over the entire distance and gets the work.

It is like a triangle with base equal to displacement and height equal to stress. The area of this triangle is $\frac{1}{2}$stress strain.

And since stress related to strain through multiplication with the matrix D, this can be written as

$$u = \frac{1}{2}\varepsilon^T D\varepsilon \quad (8)$$

By integrating u over the entire element one can find the total strain energy U in the element.

$$U = \frac{1}{2}\int\int\int \varepsilon^T D\varepsilon dV \tag{9}$$

Finding equilibrium is then all about balancing the integral above with external forces, but the strain for every sub volume of the element is not known. Only the three nodes of the triangle element are known. This means that the strain inside the element must be approximated through interpolation between the triangles known nodes. This interpolation can be written as the folowing for displacements u,v which are displacements along the x and y axis.

$$u(x,y) = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3 \tag{10}$$

$$v(x,y) = N_1(x,y)v_1 + N_2(x,y)v_2 + N_3(x,y)v_3 \tag{11}$$

Here the N's are interpolating weight functions which scale the contribution from the three node values $u_{1,2,3}$ or $v_{1,2,3}$ based on the position x,y inside the element. Any quantity can be interpolated over the element using this method but for strain calculation it is the displacement along x and y which is interpolated.

I do not show N here since it is just basic linear interpolation over a triangle.

Strain along one axis is displacement differentiated over that axis or... difference in displacement, meaning that if both ends of an element are displaced by the same amount, this gives no strain and no stress, but if one ene is displaced a different distance then the element will be strained. The more differently they are displaced, the more strain. If displacement along one axis varies along another axis, for xample x along y, $\frac{\partial x}{\partial y} \neq 0$ then there is shear strain as well.

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{12}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{13}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{14}$$

$$\tag{15}$$

The displacement inside the element is unknown so this can be rewritten as interpolated values based on the interpolation functions N and the node displacements shown here only for x strain.

$$\varepsilon_x = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \tag{16}$$

If the total element displacement is written as one vector δ then the interpolation and the stress strain relation can be written as one large matrix equation. If further more B is introduced as the strain-displacement matrix which is the partial derivatives of the interpolation functions, then it becomes

$$\varepsilon \quad = \quad \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = B\delta \tag{17}$$

In the above, stress is given from only the displacement matrix and the constant matrix B. The interpolation functions N are linear meaning that the interpolated values will be linearly interpolated over the entire triangular element. Since strain is the first derivative of displacement, which was linear, the *strain will be constant over the element*.

(9) can now be rewritten to use the total nodal displacement δ

$$U = \frac{1}{2}\int\int\int \delta^T BDB\delta dV \tag{18}$$

Since the strain is constant over the entire element, the integration can be simplified more by moving the constant part out of the integral giving

$$U = \frac{1}{2}\delta^T BDB\delta \int\int\int dV \tag{19}$$

The remaining volume integral over dV is simply the volume of the element, which means that the total strain energy U inside an element now can be written as

$$U = \frac{1}{2}\delta^T BDBV \tag{20}$$

Work is force multiplied with distance over which the force works. If the external nodal forces f are written in vector form, then the work done by those forces on the three nodes of the triangle can be written as

$$W = \delta^T f \tag{21}$$

!TODO: This force f has not been working over the entire distance. It starts out just moving into a soft object and only at the end does it push full force. This is just like the elastic strain energy previously. Should it not then be $\frac{1}{2}$f multiplied with delta? Should the above equation not be half of what is written?!

where each displacement component is multiplied with the force along the displacement.

The potential energy Π inside an element is the strain energy minus the work done by external forces

$$\Pi = U - W \tag{22}$$

Looking at how total strain energy is defined in (19), this can be expanded to

$$\Pi = \frac{1}{2}\delta^T BDBV - \delta^T f \tag{23}$$

The solution to minimal potential energy gives equilibrium, but there may in fact be local minima which are not stable. Such a configuration will, given even the slightest pertubation, fall into a state of even lower potential energy. In solving the above, it is assumed that there is only one global minimum and therefore the gradient of potential energy will be zero only for that solution. This means that solving for $\frac{\partial \Pi}{\partial \delta} = 0$ will give the sought after displacement which defines equilibrium.

Solving for the entire displacement vector δ can be done by solving for the 6 node displacements in x and y. If (23) is differentiated over δ the result for equilibrium, $\Pi = 0$ is

$$VB^T DB\delta - f = 0 \tag{24}$$

$$VB^T DB\delta = f \tag{25}$$

If the element stifness matrix k is introduced as

$$k = VB^T DB \tag{26}$$

then the solution to equilibrium, where the gradient of potential energy is zero, is

$$k\delta = f \tag{27}$$

which can easily be solved for displacements δ given the known k and f.