Finite Volume Method

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February 2, 2010

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1 Finite volume

The main idea behind the Finite Volume Method FVM is that what goes in must come out and vice-versa.

If the domain is divided into a number of volumes, or areas for 2D or lines for 1D, which are bounded by surfaces (lines in 2D and points in 1D) that are shared between pairs of such sub volumes, then it is obvious that if something leaves one volume through a shared surface then what just left will have gone into the neighbor volume through that same shared surface.

The exception to this is the outer boundaries which are not shared between sub volumes, but only belongs to one volume. At the boundaries, there need to be boundary conditions as usual.

1.1 Divergence theorem

The divergence theorem states that the outward flux from any such volume will be equal to the integral of the flux inside the cell. If the cell size is taken to the limit of infinitely small then this seem obvious since the flux in the cell and the flux on the boundary is the same thing. For larger volumes it also seems logical since the sum of every source and drain inside the volume will be the same as the flow out of the volume. This does require that the quantity flowing is conserved, does not change over time in itself, and that it is incompressible which would otherwise allow a local buildup.

A less wordy version of this is (1), where the volume integral of a the divergence of a vector field is equal to the surface integral, over the bounding surface, of the field leaving the volume. Leaving is defined as a positive dot product of the (vector) field and the outward surface normal. Moving right when right is out would mean that one is leaving and moving left when left is out would also mean that the field is leaving.

$$\int_V (\nabla \cdot F) dV = \int_S F \cdot n dS \tag{1}$$

1.2 Conservation

If nothing can leave its current volume without entering a neighbor volume then this perfectly ensures that nothing is lost and the quantity in question will be conserved. The exception is again the outer boundaries where the conditions can be defined to let things flow in and out, but this is well defined any overall positive of negative flux will be the result of design and not of errors.

2 Simplification

If the boundary is piece-wise continuous then the integral $\int_S F \cdot dS$ can be replaced with a sum over the different pieces of the boundary $\sum_e \int_{S_e} F \cdot dS$. If the pieces are furthermore linear then the integral becomes trivial since the normal is constant and we can write the right hand side, the one describing edge flux as (2).

$$\sum_e F_e \cdot n l_e \tag{2}$$

where l_e is the length of the edge - or the area of a surface in 3D.

3 Application

A simple PDE is written on integral form and then solved using FVM.

$$\int_V \frac{\partial f(u)}{\partial t} dV = \int_S f(u) \cdot n dS \tag{3}$$

This again states that the change over time of f(u) is equal to the flux over the boundary. If f(u) is entering the volume, then f(u) inside the volume increases. Here f(u) the flux function of u and f(u) is a n dimensional vector for a n dimensional space and u can be anything. It might be temperature and then f(u) is the temperature flux or the movement of thermal energy.

The problem is then to find the development of u over time given some initial value.

For this to be solved, we need to find out whet the right hand side is. Assuming a domain as in the slides with triangles sharing edges in pairs and with scalar values in the centers and vector values in the center of the edges, the right hand side can be rewritten following the reasoning from before, since the boundary around a cell is simply a line.

The left hand side can also be transformed.

$\int_V \frac{\partial f(u)}{\partial t} dV$ is the same as $\int_V \frac{\partial}{\partial t} f(u) dV$ which the differentiation operator written for itself. It can then be moved out from the integration resulting in a new left hand side as seen in (5) where the previously transformed right hand side is also included.

$$\frac{\partial}{\partial t} \int_V f(u) dV = \sum_e f(u)_e \cdot n l_e \tag{4}$$

Here $f(u)_e$ is the mean value of f(u) over the edge e. Now it is read as the rate of change over time of the flux function inside the entire volume is equal to whatever moves over the (three) linear edges. What moves through the volume is what moves through the boundaries.

Changing the left hand side to also use the mean value rather than the actual variable f(u), then it becomes

$$\frac{\partial f(u)_c V}{\partial t} = \sum_e f(u)_e \cdot n l_e \tag{5}$$

3.1 Domain subdivision grid

The volumes can have any shape you would like. The only real requirement (i believe it is) is that neighboring sub volumes should share the sane boundary edges. In other words, it is a one to one relationship on the edges. One edge does in one volume does not connect to two smaller edges of another volume. It would not be a "‘common edge"’ then...

The accuracy of the midpoint approximations depends on the relationship between surface length/area and area/volume so a very thin triangle is not as optimal as one with sides of equal length.

The problem with zero energy mode, as also known in the regular grid Finite Difference Method is also existing in FVM, when using finite difference approximations of the derivatives, so a staggered grid can also be used here. One example would be to store scalars in the center of the volumes and their derivative vectors at the boundaries. For example density in the center and velocity at the boundary. The velocity moves the densities across the boundaries.

3.2 Values outside where they are stored

With scalars stored at volume centers, one has to calculate, approximate/interpolate, their value at the edges when they are needed there. One method of finding the edge value of some quantity, stored at the centers of the volumes sharing the edge, is to calculate the average value. Another is to use an upwind scheme, or down wind if moving back in time, where we say that if the movement of the field transporting the scalar goes from volume A towards volume B, then the value at the edge is more A than B or even entirely A. It may not be true at any one point in time if the field is constant, but over the duration of a time step it will certainly be more and more correct.

3.3 Fluid simulation

The simulation of a conserved fluid flow is described by convection, diffusion and pressure which are interrelated.

Diffusion The diffusion over en edge D_e can be written, in FVM terms as (6). There is a major problem though.

$$D_e = \mu(\frac{\partial^2 u_{e,j}}{\partial x_j^2} + \frac{\partial^2 u_{e,i}}{\partial x_i^2})n_e l_e \tag{6}$$

To use the equation above, we need to find how u changes in space. This can be done by considering a linear interpolation of u along the line connecting the two relevant volume centers. In this case c and d. Such an interpolation over a variable t going from 0 when on d and 1 when on c can be written as $tu_d + (1-t)u_c$. This means that we can write the change of u over x based on the change of x over t where the latter depends only on the distance between the center positions of the volumes and the slope of the connecting line.

$$\frac{\partial u_e}{\partial x_i} = \frac{\partial}{\partial x_i}(tu_d + (1-t)u_c) \tag{7}$$

$$\frac{\partial u_e}{\partial x_i} = \frac{\partial}{\partial x_i}(tu_d + (1-t)u_c) \tag{8}$$

Now differentiate with t and multiply with t to rewrite into

$$\frac{\partial u_e}{\partial x_i} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x_i}(tu_d + (1-t)u_c) = \frac{u_d - u_c}{x_{d.i} - x_{c,i}} \tag{9}$$

The same can be repeated for the other dimension(s).

Pressure The pressure is taken as the average pressure at the edges endpoints. Pressure is a scalar and is stored at vertices which means the edge has a pressure at each end.

Advection The advection of a quantity over an edge is given by the velocity field at the edge and the concentration of the quantity there. For density this is the mass flux.

$$\rho_e u_e \cdot n_e l_e \tag{10}$$